§1.1 Field Theory

The path integral representation

$$Z := \langle 0| e^{-iHT} |0\rangle = \int \mathcal{D}q(t)\, e^{i\int_0^T dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]}$$

for the quantum mechanics of a single particle, can be generalized to the case of N particles:

$$H = \sum_a \frac{1}{2m_a}\hat{p}_a^2 + V(\hat{q}_1, \hat{q}_2, \dots, \hat{q}_N)$$

positions of particles, $a = 1, 2, \dots, N$

Going through the same steps as before, we get

$$Z = \langle 0| e^{-iHT} |0\rangle = \int \mathcal{D}q(t)\, e^{iS(q)}$$

where

$$\underset{\text{"action"}}{S(q)} = \int_0^T dt\left(\sum_a \frac{1}{2}m_a \dot{q}_a^2 - V(q_1, q_2, \dots, q_N)\right)$$

includes interaction energy $v(q_a - q_b)$ as well as external potential $w(q_a)$

We can expand

$$V(q_1, q_2, \cdots, q_N) = \sum_{a,b} \frac{1}{2} k_{ab} q_a q_b + \cdots$$

Taking the continuum limit, the label a on the particles can be replaced by n-dimensional position vector $\vec{x} \longrightarrow q_a(t) \longrightarrow q(t, \vec{x})$

Traditional notation: $q(t, \vec{x}) \longrightarrow \varphi(t, \vec{x})$ "field"

kinetic energy

$$\sum_a \frac{1}{2} m_a \dot{q}_a^2 \longrightarrow \int d^n x \, \frac{1}{2} \sigma \left(\partial \varphi / \partial t\right)^2$$

where $\sum_a \longrightarrow \frac{1}{\ell^n} \int d^n x$, $\sigma := \frac{m_a}{\ell^n}$

characteristic length scale

take m_a's equal

Focusing on $V = \sum_{a,b} \frac{1}{2} k_{ab} q_a q_b + \cdots$, and using $2 q_a q_b = (q_a - q_b)^2 - q_a^2 - q_b^2$, we get for nearest-neighbor interactions,

$$(q_a - q_b)^2 \sim \ell^2 (\vec{\nabla}\varphi)^2$$

$$\longrightarrow \quad S(\varphi) := \int_0^T dt \int d^n x\, \mathcal{L}(\varphi)$$

$$= \int_0^T dt \int d^n x\, \frac{1}{2}\left(\sigma\left(\frac{\partial\varphi}{\partial t}\right)^2 - \rho\left[\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \cdots \left(\frac{\partial\varphi}{\partial x_n}\right)^2\right] - \tau\varphi^2 - \nu\varphi^4 + \cdots\right)$$

where parameters $\rho, \tau, \ldots$ are determined by $K_{ab}, \ell, \ldots$

This already looks almost like a Lorentz invariant action!

$\longrightarrow$ In QFT we impose Lorentz invariance and no more than two time derivatives

$\longrightarrow$ set $\rho = \sigma c^2$, $\varphi \mapsto \frac{\varphi}{\sqrt{\sigma}}$

$\longrightarrow$ obtain Kinetic term

$$(\partial\varphi/\partial t)^2 - c^2\left[(\partial\varphi/\partial x)^2 + (\partial\varphi/\partial y)^2\right]^2$$

Setting $c=1$ and $D=n+1$, we end up with a Lorentz-inv. action:

$$S = \int d^D x \left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 + \dots \right]$$

Notation: $(\partial\varphi)^2 := \partial_\mu \varphi \, \partial^\mu \varphi$

Note: Higher spacial derivatives are naturally excluded due to Lorentz-invariance!

Insisting on further symmetries, like $\varphi \mapsto -\varphi$, has implications on $V(\varphi)$ (e.g. only even polynomials)

Classical limit:

Restoring $\hbar$ gives

$$Z = \int \mathcal{D}\varphi \, e^{(i/\hbar)\int d^D x \, \mathcal{L}(\varphi)}$$

In the limit $\hbar \to 0$, one obtains according to Euler-Lagrange variation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\delta(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

§1.2 The operator formalism and the vacuum

Review of Heisenberg's approach to QM:

Consider $\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q)$

single particle Lagrangian

→ define canonical momentum

$$p := \frac{\delta \mathcal{L}}{\delta \dot{q}} = \dot{q}$$

$$\rightarrow \quad H = p\dot{q} - \mathcal{L} = \frac{p^2}{2} + V(q)$$

p and q become operators by imposing

$$[\hat{p}, \hat{q}] = -i \quad (*) \qquad (\text{setting } \hbar = 1)$$

time evolution:

- $\frac{dp}{dt} = i[H, p] = -V'(q)$
- $\frac{dq}{dt} = i[H, q] = p$

→ operators evolve according to

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$$

Defining operator

$$a := \frac{1}{\sqrt{2\omega}}(\omega q + ip)$$

with some parameter ω, we get

$$\xrightarrow{(*)} [a, a^\dagger] = 1$$

$a(t)$ evolves according to

$$\frac{da}{dt} = i\left[H, \frac{1}{\sqrt{2\omega}}(\omega q + ip)\right]$$

$$= -i\sqrt{\frac{\omega}{2}}\left(ip + \frac{1}{\omega}V'(q)\right)$$

$\longrightarrow$ ground state is defined as

$$a|0\rangle = 0$$

For the harmonic oscillator we get

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \longrightarrow V'(q) = \omega^2 q$$

$$\longrightarrow \frac{da}{dt} = -i\omega a$$

Generalization to field theory:

$$\mathcal{L} = \int d^{D-1}x \left[\frac{1}{2}\left(\dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 - m^2\varphi^2\right) - U(\varphi)\right]$$

anharmonic term

→ canonical momentum density:

$$\pi(\vec{x},t) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(\vec{x},t)} = \partial_0 \varphi(\vec{x},t)$$

canonical commutation relations at equal time:

$$[\pi(\vec{x},t), \varphi(\vec{x}',t)] = [\partial_0 \varphi(\vec{x},t), \varphi(\vec{x}',t)]$$
$$= -i\,\delta^{(D-1)}(\vec{x}-\vec{x}')$$

and $[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$,

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$$

→ Hamiltonian is given by Legendre transformation:

$$H = \int d^{D-1}x \, [\pi(\vec{x},t)\,\partial_0\varphi(\vec{x},t) - \mathcal{L}]$$
$$= \int d^{D-1}x \left[\frac{1}{2}\left(\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2\right) + u(\varphi) \right]$$

For the case of the harmonic oscillator ($u=0$), the field equation is:

$$(\partial^2 + m^2)\,\varphi = 0 \qquad (*)$$

Define operators $a(\vec{k})$, $a^\dagger(\vec{k})$, with

$$[a(\vec{k}), a^\dagger(\vec{k})] = \delta^{(D-1)}(\vec{k}-\vec{k}') \qquad (**)$$

$\longrightarrow$ solution to (*) :

$$\varphi(\vec{x},t) = \int \frac{d^{D-1}k}{\sqrt{(2\pi)^{D-1} 2\omega_k}} \left[a(\vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a^\dagger(\vec{k}) e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

with $\omega_{\vec{k}} = + \sqrt{\vec{k}^2 + m^2}$

$\longrightarrow$ $[\partial_0 \varphi(\vec{x},t), \varphi(\vec{x}', t)] = -i\,\delta^{(D-1)}(\vec{x}' - \vec{x}')$

automatically satisfied !

Now we are ready to defining the vacuum !

$\longrightarrow$ It is the ground state of QFT:

$$a(\vec{k})\,|0\rangle = 0 \quad \forall\ \vec{k}$$

<u>Energy of vacuum</u> :

We want to compute the expectation value:

$$E_0 := \langle 0|\, H \,|0\rangle$$
$$= \frac{1}{2} \int d^{D-1}x\, \langle 0|\, \pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2 \,|0\rangle$$

Focusing on the last term, we have

$$\langle 0|\varphi(\vec{x},t)\,\varphi(\vec{x},t)|0\rangle \overset{\text{translation invariance}}{=} \langle 0|\varphi(\vec{0},0)\,\varphi(\vec{0},0)|0\rangle$$

$$= \lim_{\vec{x},t\to\vec{0},0} \langle 0|\varphi(\vec{x},t)\,\varphi(\vec{0},0)|0\rangle$$

$\rightarrow$ thus we need to compute the two-point correlator

$$\langle 0|\varphi(\vec{x},t)\,\varphi(\vec{0},0)|0\rangle \quad \text{for } t>0$$

Of the four terms $a^\dagger a^\dagger$, $a^\dagger a$, $a a^\dagger$, and aa in the above product, only the $aa^\dagger$ survives

$\rightarrow$ using commutator (**), we get

$$\langle 0|\varphi(\vec{x},t)\,\varphi(\vec{0},0)|0\rangle$$

$$= \int \frac{d^{D-1}k}{(2\pi)^{D-1}\,2\omega_k}\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}$$

More generally, defining the time-ordered product,

$$T[\varphi(x)\varphi(y)] = \Theta(x^0-y^0)\,\varphi(x)\varphi(y) + \Theta(y^0-x^0)\,\varphi(y)\varphi(x),$$

we get

$$\langle 0|T[\varphi(\vec{x},t)\varphi(\vec{0},0)]|0\rangle$$

$$= \int \frac{d^{D-1}k}{(2\pi)^{D-1}\,2\omega_k}\left[\Theta(t)\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \Theta(-t)\, e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}\right]$$

Putting everything together, we compute

$$\langle 0| H |0\rangle = \frac{1}{2}\int d^{D-1}x \, \langle 0| (\partial_t \varphi)^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2 |0\rangle$$

$$= vol_{D-1} \int \frac{d^{D-1}k}{(2\pi)^{D-1} 2\omega_k} \left[\frac{1}{2}\left(\omega_k^2 + \vec{k}^2 + m^2\right)\right]$$

volume of space

$$= vol_{D-1} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{2} \hbar \omega_k$$

where in the last line we restored the dependence on $\hbar$

$\longrightarrow$ redefine Hamiltonian to

$$H - \langle 0|H|0\rangle$$

such that the vacuum energy is zero!